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DE RHAM HOMOLOGY FOR NETWORKS OF MANIFOLDS

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ABSTRACT

This paper describes in terms of differential forms the real homology of a certain class of spaces, which we call networks. Networks include, besides smooth manifolds, singular sets of toral actions, classifying spaces of Lie groups, etc. A generalized Thom isomorphism theorem is also proved in this context.

Introduction

The purpose of this paper is to describe in terms of differential forms the real homology of a certain class of spaces, which we call networks, and to prove a generalized Thom isomorphism theorem in this context.

Networks are spaces which can be decomposed as unions of smooth manifolds satisfying some specific conditions. The precise definition is given in §1.

The main example of a network is provided by the "singular set" of a toral action. In fact this is the example that has motivated our definition. We describe it next.

Let G be a torus, or more generally an abelian compact Lie group, acting smoothly on a smooth manifold M. Then the set A of points in M left fixed by some nontrivial subtorus of G, as well as the set A' of points in M left fixed by some nontrivial subgroup of G, are networks if we further assume that the action of G on M has finite orbit type (see theorem 14 of [4]).

If a network A were a compact oriented m-manifold we could use the exterior derivative in the differential forms on A and write $A_p(A) = A^{m-p}(A)$; this would give the homology of A (by Poincaré duality). In general A is not a smooth manifold, thus we need to work a little harder and construct a chain complex out of differential forms giving the real homology of A. This construction is carried out in §2.

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In §3 we establish an isomorphism between the de Rham cohomology with compact supports of some tubular neighbourhood of a network A, embedded in a manifold M, and the real homology of A. This isomorphism reduces to the Thom isomorphism when A is a smooth manifold.

We prove in §4 that the homology constructed in §2 is, in fact, isomorphic to the real singular homology of A.

Finally in §5 we compare networks with other constructions such as Mostow's differentiable space structure, see [6], and study more closely the particular case of a smooth toral action.

Next I wish to describe the kind of applications I have in mind in writing this paper. Suppose we can define some sort of structure on a compact oriented smooth manifold M, but only outside some "singular set" A, which happens to be a network. Further assume that for any given open neighbourhood U of A it is possible to obtain differential forms representing some characteristic classes α of a vector bundle ξ over M and these differential forms having compact support contained in U. In this situation we produce a real homology class of A, β, by using the generalized Thom isomorphism theorem (13). Thus we have a formula $j_*\beta = D\alpha$, where $j : A \to M$ is the inclusion and D is Poincaré duality. In many cases we can think of β as a kind of "residue" at A which is obtained in terms of local data. The above formula would be thus a residue formula relating global and local information.

In a forthcoming paper I will show one of these residue formulae for the particular case of a 2-dimensional torus acting smoothly on a vector bundle. Nevertheless I believe Theorem 13 of this paper could possibly be used in similar situations.

Notation

We use the word manifold to mean a disjoint union of connected Hausdorff C^∞-manifolds with a countable base of open sets.

We do not assume that the connected components of a manifold have all the same dimension.

By a submanifold N of a manifold M we mean a manifold N topologically embedded in M such that each connected component of N is a smooth submanifold of some connected component of M.

If $B = \bigcup_{i \in I} B_i$ is a manifold with connected components $\{B_i\}_{i \in I}$, we set $A^p(B) = \Pi_{i \in I} A^p(B_i)$, where $A^p(B_i)$ are the spaces of exterior p-forms on B_i.

$A^p_c(B) = \bigoplus_{i \in I} A^p_c(B_i)$, where $A^p_c(B_i)$ are the spaces of exterior p-forms on B_i having compact support.

$A_p(B) = \Pi_{i \in I} A^{n_i - p}(B_i)$, where n_i = dimension of B_i, with the convention that $A^{n_i - p}(B_i) = 0$ whenever $n_i - p$ is negative.

$A_p^c(B) = \bigoplus_{i \in I} A_c^{n_i - p}(B_i)$, with the same convention as before.

Similarly a fibre bundle over a manifold B is understood as a family of smooth fibre bundles, one over each B_i.

If ξ is an oriented fibre bundle over a manifold B and we denote by E its total space, we have the fibre integral $\oint : A_p^c(E) \to A_p^c(B)$ as defined in chapter 7 of [5].

§1. Networks

(1) Let I be an ordered set such that each pair i, j in I with a lower bound has an infimum (greatest lower bound), denoted $\inf(i, j)$. We call such I's *almost directed sets.*

Regard I as a category in the obvious way and define a *network* as a covariant functor from I to the category of manifolds and smooth maps, such that if we denote by F_i the manifold associated to $i \in I$ and by $\varphi_{ij} : F_i \to F_j$ the smooth map associated to $i \leqq j$, the following conditions are verified (see [4]):

(a) (F_i, φ_{ij}) is a closed submanifold of F_j $(i, j \in I, i \leqq j)$.

(b) $\varphi_{jr}(F_j) \cap \varphi_{kr}(F_k) = \varphi_{ir}(F_i)$, $i, j, k, r \in I$ with $j \leqq r$, $k \leqq r$ and $i = \inf(j, k)$.

(c) $T_x(\varphi_{ir}(F_i)) = T_x(\varphi_{jr}(F_j)) \cap T_x(\varphi_{kr}(F_k))$ for all $x \in \varphi_{ir}(F_i)$ and i, j, k, r as in (b).

(d) For all $r \in I$ and $x \in \bigcup_{i \leq r} \varphi_{ir}(F_i)$, the family of subspaces of $T_x(F_r) \{T_x(\varphi_{ir}(F_i))\}_{i \leq r}$ verifies

$$T_x(\varphi_{ir}(F_i)) \cap \sum_{j \in J} T_x(\varphi_{jr}(F_j)) = \sum_{j \in J} T_x(\varphi_{ir}(F_i)) \cap T_x(\varphi_{jr}(F_j))$$

for $i \in I_{\leq r}$, $J \subset I_{\leq r}$ (where $I_{\leq r} = \{i \in I \mid i \leqq r\}$).

(2) If $\mathcal{F} = (\{F_i\}_{i \in I}, \varphi_{ij})$, $\mathcal{F}' = (\{F'_i\}_{i \in I'}, \varphi'_{ij})$ are networks, a smooth map $f : \mathcal{F} \to \mathcal{F}'$ consists of a nondecreasing map $\alpha : I \to I'$ together with a family of smooth maps $f_i : F_i \to F'_{\alpha(i)}$ such that the diagrams

$$\begin{array}{ccc} F_i & \xrightarrow{f_i} & F'_{\alpha(i)} \\ \downarrow{\scriptstyle \varphi_{ij}} & & \downarrow{\scriptstyle \varphi'_{\alpha(i)\alpha(j)}} \\ F_j & \xrightarrow[f_j]{} & F'_{\alpha(j)} \end{array}$$

commute for all $i \leqq j$ in I.

(3) Observe that if $\mathcal{F}$ is a network we can define the following topological space $|\mathcal{F}|$: consider the disjoint union of the F_i ($i \in I$) and identify $x \in F_i$ with $y \in F_j$ if there exists $k \in I$ such that $k \leqq i$, $k \leqq j$ and $\varphi_{ki}(z) = x$, $\varphi_{kj}(z) = y$ for some $x \in F_k$.

We define a topology on $|\mathcal{F}|$ as the finest topology for which all natural inclusions $\varphi_i : F_i \to |\mathcal{F}|$ are continuous. Thus, a map $f : |\mathcal{F}| \to X$ (X being any topological space) is continuous if and only if $f \circ \varphi_i : F_i \to X$ is continuous for all $i \in I$.

Clearly a smooth map, $f : \mathcal{F} \to \mathcal{F}'$, of networks induces a continuous map $|f| : |\mathcal{F}| \to |\mathcal{F}'|$.

Observe that the topology induced from $|\mathcal{F}|$ to $\varphi_i(F_i)$ coincides with the topology induced from F_i via φ_i. Therefore we may assume that all maps φ_{ij}, φ_i are inclusions and often we do not write them.

(4) *Examples of Networks*

(a) *Manifolds.* A manifold can be regarded as a network over the almost directed set consisting of a single element.

(b) *Wedge of manifolds.* A wedge of manifolds $\vee_{i \in I} M_i$ is a network over $I \cup \{*\}$ ($*$ is the common point of the M_i) with $* < i$ for all $i \in I$ and no other pair of distinct elements being related.

(c) *Fixed point sets of toral actions* (see [4]). This is the main example that has motivated the theory.

Let G be a torus (or more generally a compact abelian Lie group) acting smoothly on a compact manifold M, then the sets $F = \{x \in M \mid G_x \neq e\}$ and $F^0 = \{x \in M \mid G_x^0 \neq e\}$ can be expressed as the union of submanifolds of M. Here e denotes the unit element of G, G_x denotes the isotropy subgroup at x of the action of G on M and G_x^0 denotes the 1-component of G_x.

(d) *Classifying spaces of Lie groups.* In fact these spaces can be described as the union of a sequence of manifolds $\cdots M_i \subset M_{i+1} \cdots$ with M_i a closed submanifold of M_{i+1}.

(e) *Network associated to a simplicial complex.* Let K be a simplicial complex and denote by I_K the set of all simplices of K ordered by inclusion. I_K is an almost directed set. To each $\sigma \in I_K$ we associate the linear manifold F_σ of all functions $\alpha : K \to R$ such that $\alpha(v) = 0$ if $v \notin \sigma$ and $\Sigma_{v \in \sigma} \alpha(v) = 1$. Therefore $\mathcal{F} = \{F_\sigma\}_{\sigma \in I_K}$ is a network associated to K. The polyhedron $|K|$ is a closed subspace of $|\mathcal{F}|$. Furthermore, $|K|$ is a strong deformation retract of $|\mathcal{F}|$.

(f) *Linear network.* Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a family of distinct finite dimensional subspaces of a real vector space F of arbitrary dimension and assume that the following two conditions hold:

(i) The intersection of any two members of $\mathcal{F}$ belongs to $\mathcal{F}$.

(ii) $F_i \cap (\Sigma_{j\in J} F_j) = \Sigma_{j\in J} F_i \cap F_j$ for all $i \in I$ and $J \subset I$.

Then $\mathcal{F}$ is clearly a network.

(g) *Standard network.* Let $\xi : E \xrightarrow{\pi} B$ be a real vector bundle and let $\xi_i : E_i \xrightarrow{\pi_i} B$ $(i = 1, \dots, r)$ be subbundles of ξ such that $\xi = \xi_1 \oplus \cdots \oplus \xi_r$.

If $I \subset \{1, \dots, r\}$ we denote $\bigoplus_{i\in I} \xi_i$ by $\xi^I : E_I \xrightarrow{\pi^I} B$ and set $\xi^{\varnothing} = 0$ — vector bundle over B, so that we identify $E_{\varnothing}$ with B.

The family $\{E_I\}_{I\subset\{1,\dots,r\}}$ is a network in E as can be checked easily. We call it the standard network associated to $(\xi; \xi_1, \dots, \xi_r)$.

(5) *Orientation of a Network*

We say that a network $\mathcal{F} = \{F_i\}_{i\in I}$ is *orientable* if each of the manifolds F_i is. An orientation of $\mathcal{F}$ is then a particular choice of orientations, one for each F_i. Of course we assume that in case F_i and F_j have a common connected component, both orientations agree on it.

We have for the above examples (4):

(a) Of course a manifold is oriented as a network if it is oriented as a manifold.

(b) $\vee_{i\in I} M_i$ is oriented if each M_i is.

(c) F and F^0 can be expressed as the union of the members of oriented networks if M is oriented.

(d) The classifying space of a compact connected Lie group is oriented.

(e) and (f) The network associated to a simplicial complex as well as any linear network is always oriented.

(g) The standard network is oriented if B is oriented and each ξ_i is also oriented.

If a network $\mathcal{F}$ has been oriented with an orientation $\mathcal{O}$, we will denote $\mathcal{F}_{\mathcal{O}}$ the network $\mathcal{F}$ with the given orientation.

(6) *Tubular Neighbourhoods*

Let $\mathcal{F} = \{F_i\}_{i\in I}$ be a network. We define an internal tubular neighbourhood of $\mathcal{F}$ to be a family of tubular neighbourhoods $\rho_{ij} : U_{ij} \to F_i$ of F_i in F_j $(i \leqq j)$ verifying the following conditions:

(i) If $i \leqq r$, $j \leqq s$ and $F_i \cap F_j = \varnothing$ then $U_{ir} \cap U_{js} = \varnothing$.

(ii) If $i \leqq j \leqq k$ then $\rho_{jk}^{-1}(U_{ij}) = U_{ik}$ and $\rho_{ij} \circ \rho_{jk}\big|_{U_{ik}} = \rho_{ik}$.

(iii) If $j \leqq r$, $k \leqq r$, $F_j \cap F_k = F_i$ then we have $U_{ij} = F_j \cap U_{kr}$ and $\rho_{ij} = \rho_{kr}\big|_{U_{ij}}$.

(iv) If $j \leqq r$, $k \leqq r$ and $F_j \cap F_k = F_i$ then $U_{jr} \cap U_{kr} = U_{ir}$.

(v) If $j \leqq r$, $k \leqq r$ and $F_j \cap F_k = F_i$ then $\bar{U}_{jr} \cap \bar{U}_{kr} = \bar{U}_{ir}$, where all closures are taken in F_r.

REMARK. If F is a submanifold of M, an internal tubular neighbourhood of F is simply $F \xrightarrow{\text{id}} F$, and an internal tubular neighbourhood of the network $\{F, M\}$ consists of the three tubular neighbourhoods $F \xrightarrow{\text{id}} F$, $U \xrightarrow{\rho} F$, $M \xrightarrow{\text{id}} M$. Of course, we normally say simply that $U \xrightarrow{\rho} F$ is an internal tubular neighbourhood of $\{F, M\}$.

(7) The examples (4) a, b, d and f admit clearly an internal tubular neighbourhood. The example (g) admits the internal tubular neighbourhood $\pi_I^J : E_J \to E_I$ given by $\pi_I^J(\bigoplus_{i \in J} z_i) = \bigoplus_{i \in I} z_i$ $(I \subset J)$. The examples c and e admit also an internal tubular neighbourhood if the index set is finite. In fact one can prove, by standard differential geometric arguments, the existence of internal tubular neighbourhoods for any finite network. We wish to emphasize that conditions (c) and (d) of the definition of a network are needed here to prove the existence of such an internal tubular neighbourhood.

§2. De Rham homology of an oriented network

Let $\mathscr{F} = \{F_i\}_{i \in I}$ be an oriented network admitting an internal tubular neighbourhood $\mathscr{T} = \{(U_{ij}, \rho_{ij})\}_{i \leqq j}$.

If $F_i \subset F_j$ we have on U_{ij} the orientation induced from F_j, and we endow then the vector bundle $\rho_{ij} : U_{ij} \to F_i$ with the unique orientation such that the orientation on U_{ij} coincides with the local product orientation of the one of F_i and the one of the vector bundle (see [5]). We say that $\mathscr{T}$ has been oriented with the induced orientation.

(8) Let $\mathscr{O}$ be an orientation of a network $\mathscr{F} = \{F_i\}_{i \in I}$ and let us denote the oriented network by $\mathscr{F}_{\mathscr{O}}$.

Consider the real vector space $\bigoplus_{i \in I} A_p^c(F_i)$ and define $J_p^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T})$ as the subspace spanned by the elements of the form

$$\left(\Phi_j, -\int_{ij} \Phi_j\right) \in A_p^c(F_j) \oplus A_p^c(F_i), \qquad i \leqq j,$$

where Φ_j has compact support on U_{ij}.

Set $A_p^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T}) = \bigoplus_{i \in I} A_p^c(F_i) / J_p^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T})$.

$J_*^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T}) = \bigoplus_{p \geqq 0} J_p^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T})$ is stable under the exterior derivative d and hence we have an induced map $\bar{d} : A_p^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T}) \to A_{p-1}^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T})$.

We denote by $H_*^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T})$ the homology of the chain complex $(A_*^c(\mathscr{F}_{\mathscr{O}}, \mathscr{T}), \bar{d})$.

$H^c_*(\mathcal{F}_\mathcal{O}, \mathcal{T})$ is called the *de Rham homology of the oriented network* $\mathcal{F}$. We will see in §4 that this homology is canonically isomorphic to the singular homology of $|\mathcal{F}|$ with real coefficients. In particular it will be shown that it is independent of the particular choice of $\mathcal{T}$ and $\mathcal{O}$.

(9) Similarly if we consider $\bigoplus_{i\in I} A_p(F_i)$ and define $J_p(\mathcal{F}_\mathcal{O}, \mathcal{T})$ as the span of

$$\left(\Phi_j, -\int_{ij} \Phi_j\right) \in A_p(F_j) \oplus A_p(F_i), \qquad i \leqq j,$$

with Φ_j having fibre compact support on U_{ij} (see [5]), we obtain another chain complex $(A_*(\mathcal{F}_\mathcal{O}, \mathcal{T}), \bar{d})$ and hence a homology $H_*(\mathcal{F}_\mathcal{O}, \mathcal{T})$. Of course $H_*(\mathcal{F}_\mathcal{O}, \mathcal{T})$ coincides with $H^c_*(\mathcal{F}_\mathcal{O}, \mathcal{T})$ if all F_i are compact with the possible exception of the F_i where i is a maximal element of I.

(10) *Remark*

Suppose that $\mathcal{O}$ and $\mathcal{O}'$ are two orientations on $\mathcal{F}$. Define then an isomorphism

$$\bigoplus_{i\in I} A^c_p(F_i) \to \bigoplus_{i\in I} A^c_p(F_i)$$

sending $\Phi_i \in A^c_p(F_i)$ to Φ_i or $-\Phi_i$ depending on whether the orientations coincide or not on F_i.

Clearly the above isomorphism sends $J^c_p(\mathcal{F}_\mathcal{O}, \mathcal{T})$ to $J^c_p(\mathcal{F}_{\mathcal{O}'}, \mathcal{T})$ and hence induces an isomorphism from $H^c_p(\mathcal{F}_\mathcal{O}, \mathcal{T})$ onto $H^c_p(\mathcal{F}_{\mathcal{O}'}, \mathcal{T})$.

Similarly we obtain an isomorphism from $H_p(\mathcal{F}_\mathcal{O}, \mathcal{T})$ onto $H_p(\mathcal{F}_{\mathcal{O}'}, \mathcal{T})$.

We will normally drop the reference to the orientation on $\mathcal{F}$ because of this remark.

§3. Generalized Thom isomorphism theorem

Let $\mathcal{F} = \{F_i\}_{i\in I}$ be an oriented network and assume that I is finite and has a maximum $\infty \in I$ (this is not a real restriction since any finite network can be embedded in some convenient R^N). Denote F_∞ by M. Let $\mathcal{F}'$ be the network obtained by deleting M from $\mathcal{F}$. Assume that $\mathcal{T}$ is an internal tubular neighbourhood of $\mathcal{F}$ and denote by $\mathcal{T}'$ the corresponding internal tubular neighbourhood of $\mathcal{F}'$, the restriction of $\mathcal{T}$. Set $\mathcal{F}'' = \mathcal{F} \cup \{U\}$, where $U = \bigcup_{i\in I} U_i$ $(U_i = U_{i\infty})$ and $\mathcal{T}'' = (\mathcal{T} - \{M\}) \cup \{U\} = \mathcal{T}' \cup \{U_i\}_{i\in I} \cup \{U\}$ (it is an internal tubular neighbourhood of $\mathcal{F}''$).

Define

$$\rho : A^c_p(U) \to A^c_p(\mathcal{F}', \mathcal{T}')$$

as follows:

If $\Phi \in A_p^c(U)$, choose $\Phi_i \in A_p^c(U_i)$ for all $i \in I$ such that $\Phi = \Sigma_{i\in I}\Phi_i$ (this can be done easily by using a smooth partition of unity subordinate to the open covering $\{U_i\}_{i\in I}$ of U).

Define then

$$\rho(\Phi) = \left[\left(\textstyle\oint_i \Phi_i\right)_{i\in I}\right] \in A_p^c(\mathcal{F}', \mathcal{T}')$$

where [ ] denotes the corresponding class and $\oint_i$ is the fibre integral for the bundle $\rho_i : U_i \to F_i$.

Since ρ commutes with $\bar{d}$ we obtain an induced linear map

$$\rho_* : H_p^c(U) \to H_p^c(\mathcal{F}', \mathcal{T}') \tag{11}$$

where we recall that $H_p^c(U)$ means the $(n-p)$-real space of the de Rham cohomology with compact support of U.

It remains to be checked that the definition of ρ is correct, i.e., independent of the choice of Φ_i. To see this let us consider the following linear map:

$$h : A_p^c(\mathcal{F}', \mathcal{T}') \to A_p^c(\mathcal{F}'', \mathcal{T}'')$$

given by

$$h([(\Phi_i)_{i\in I}]) = [(\Phi_i)_{i\in I}, 0].$$

Then we have

$$h\left(\left[\left(\textstyle\oint_i \Phi_i\right)_{i\in I}\right]\right) = \left[\left(\textstyle\oint_i \Phi_i\right)_{i\in I}, 0\right]$$

$$= \left[\left(\textstyle\oint_i \Phi_i\right)_{i\in I}, 0\right] + \left[\left(-\textstyle\oint_i \Phi_i\right)_{i\in I}, \sum_{i\in I}\Phi_i\right] = [(0)_{i\in I}, \Phi].$$

Therefore ρ is well defined as a consequence of the following lemma:

(12) LEMMA. *The above linear map $h : A_p^c(\mathcal{F}', \mathcal{T}') \to A_p^c(\mathcal{F}'', \mathcal{T}'')$ is injective.*

PROOF. We do it in two steps labelled (a) and (b).

(a) $\alpha \in$ kernel of h, if and only if there exist $\Omega_i \in A_p^c(U_i)$ for all $i \in I$ such that $\Sigma_{i\in I}\Omega_i = 0$ and $(\oint_i \Omega_i)_{i\in I}$ represents α.

In fact, suppose we have $\Omega_i \in A_p^c(U_i)$ for all $i \in I$ and such that $\Sigma_{i\in I}\Omega_i = 0$ and let α be the class in $A_p^c(\mathcal{F}', \mathcal{T}')$ represented by $(\oint_i \Omega_i)_{i\in I}$. Thus $((\oint_i \Omega_i)_{i\in I}, -\Sigma_{i\in I}\Omega_i) \in J_p^c(\mathcal{F}'', \mathcal{T}'')$ represents $h(\alpha)$ and so $h(\alpha) = 0$.

Conversely, assume that $\alpha \in \ker h$. Choose $(\Phi_i)_{i\in I}$ to represent α and hence $((\Phi_i)_{i\in I}, 0) \in J_p^c(\mathcal{F}'', \mathcal{T}'')$. Therefore we can write in $\bigoplus_{i\in I} A_p^c(F_i) \oplus A_p^c(U)$

$$((\Phi_i)_{i\in I},0)=\Phi+\sum_{i\in I}\left(-\Omega_i+\int_i\Omega_i\right)=\Phi+\left(\left(\int_i\Omega_i\right)_{i\in I},-\sum_{i\in I}\Omega_i\right)$$

with $\Phi\in J_p^c(\mathcal{F}',\mathcal{T}')$ and $\Omega_i\in A_p^c(U_i)$ for all $i\in I$.

Define then $(\Phi_i')_{i\in I}$ representing α by

$$(\Phi_i')_{i\in I}=(\Phi_i)_{i\in I}-\Phi\in\bigoplus_{i\in I}A_p^c(F_i).$$

Then, in $\bigoplus_{i\in I}A_p^c(F_i)\oplus A_p^c(U)$, we have

$$((\Phi_i')_{i\in I},0)=\left(\int_i\Omega_i,-\sum_{i\in I}\Omega_i\right)\in J_p^c(\mathcal{F}'',\mathcal{T}'').$$

Therefore $\Phi_i'=\int_i\Omega_i$ for all $i\in I$ and $\Sigma_{i\in I}\Omega_i=0$.

(b) We prove now the injectivity of h by induction on the number of elements of I.

If I has only one element it is an immediate consequence of the characterization of the kernel of h given in (a).

Suppose the lemma holds for I with less than r elements ($r\geqq 2$) and that I has r elements $F_1,\dots,F_r$. Assume that F_r is maximal in $\mathcal{F}'$. Let $\alpha\in\ker h$ and let $(\int_i\Omega_i)_{i\in I}$ be a representative of α with $\Sigma_{i\in I}\Omega_i=0$ and $\Omega_i\in A_p^c(U_i)$ for all $i\in I$.

Set $V=U-\operatorname{support}\Omega_r$. Then $\{V,\{U_r\cap U_i\}_{i=1,\dots,r-1}\}$ is an open cover of U.

Let $\rho_1,\dots,\rho_{r-1},\rho_r$ be a smooth partition of unity subordinate to the open covering $\{\{U_r\cap U_i\}_{i=1,\dots,r-1},V\}$.

Since $\rho_r\Omega_r=0$ we have

$$\Omega_r=\sum_{i=1}^{r-1}\rho_i\Omega_r.$$

Assume that $F_1,\dots,F_\lambda$ (for some λ with $1\leqq\lambda\leqq r-1$) are all the manifolds in the family $\{F_1,\dots,F_{r-1}\}$ with nonempty intersection with F_r (the case $\lambda=0$ is considered later in the remark). Therefore $U_r\cap U_i=\varnothing$, $i=\lambda+1,\dots,r-1$ because of (6)i, and so $\rho_i\Omega_r=0$ $(i=\lambda+1,\dots,r-1)$.

Thus $\Omega_r=\Sigma_{i=1}^{\lambda}\rho_i\Omega_r$ yields $\int_r\Omega_r=\Sigma_{i=1}^{\lambda}\int_r\rho_i\Omega_r$.

Define now

$$\Omega_i'=\Omega_i+\rho_i\Omega_r,\qquad i=1,\dots,\lambda,$$

$$\Omega_i'=\Omega_i,\qquad i=\lambda+1,\dots,r-1,$$

and we have

$$\sum_{i=1}^{r-1}\Omega_i'=\sum_{i=1}^{r-1}\Omega_i+\sum_{i=1}^{\lambda}\rho_i\Omega_r=\sum_{i=1}^{r-1}\Omega_i+\left(\sum_{i=1}^{\lambda}\rho_i\right)\Omega_r=\sum_{i=1}^{r-1}\Omega_i+\Omega_r-\rho_r\Omega_r=\sum_{i=1}^{r}\Omega_i=0.$$

Next we want to show that $(\textstyle\int_1\Omega_1',\dots,\int_{r-1}\Omega_{r-1}',0_r)$ represents α, because this fact together with step (a) and induction hypothesis yields $\alpha=0$.

We have $U_i\cap U_r=U_{\inf(i,r)}$ $(i=1,\dots,\lambda)$ in view of (6)iv. We also have $\rho_i\Omega_r\in A_p^c(U_{\inf(i,r)})$.

Observe that $\int_r\rho_i\Omega_r\in A_p^c(U_{\inf(i,r)r})$. In fact we clearly have $\operatorname{supp}\int_r\rho_i\Omega_r\subset\rho_r(U_{\inf(i,r)})$, because $\operatorname{supp}\rho_i\Omega_r\subset U_{\inf(i,r)r}$. But, by (6)ii, $U_{\inf(i,r)}=\rho_r^{-1}(U_{\inf(i,r)r})$. Therefore $\operatorname{supp}\int_r\rho_i\Omega_r\subset U_{\inf(i,r)r}$.

Consider the equality

$$\int_r\rho_i\Omega_r=\int_r\rho_i\Omega_r-\int_{\inf(i,r)r}\int_r\rho_i\Omega_r+\int_{\inf(i,r)r}\int_r\rho_i\Omega_r.$$

But

$$\int_r\rho_i\Omega_r-\int_{\inf(i,r)r}\int_r\rho_i\Omega_r\in J_p^c(\mathcal{F}',\mathcal{T}')$$

and

$$\int_{\inf(i,r)r}\int_r\rho_i\Omega_r=\int_{\inf(i,r)}\rho_i\Omega_r=\int_{\inf(i,r)i}\int_i\rho_i\Omega_r\in A_p^c(F_{\inf(i,r)}).$$

Therefore

$$\int_r\rho_i\Omega_r=\int_i\rho_i\Omega_r+\Phi\qquad\text{with }\Phi\in J_p^c(\mathcal{F}',\mathcal{T}')\quad(i=1,\dots,\lambda).$$

Thus we can write

$$\begin{aligned}\sum_{i=1}^{r}\int_i\Omega_i&=\sum_{i=1}^{r-1}\int_i\Omega_i+\sum_{i=1}^{\lambda}\int_r\rho_i\Omega_r\\&=\sum_{i=1}^{r-1}\int_i\Omega_i+\sum_{i=1}^{\lambda}\int_i\rho_i\Omega_r+\Phi=\sum_{i=1}^{r-1}\int_i\Omega_i'+\Phi\end{aligned}$$

with $\Phi\in J_p^c(\mathcal{F}',\mathcal{T}')$. Hence $(\textstyle\int_1\Omega_1',\dots,\int_{r-1}\Omega_{r-1}',0_r)$ represents α.

Remark. If $F_i\cap F_r=\varnothing$ $(i=1,\dots,r-1)$, then $U_i\cap U_r=\varnothing$ by (6)i and so $\Omega_r=0$. Thus we also have that $(\textstyle\int_1\Omega_1',\dots,\int_{r-1}\Omega_{r-1}',0_r)$ is a representative of α as before, just taking $\Omega_i'=\Omega_i$ $(i=1,\dots,r-1)$.

(13) Theorem (Generalized Thom Isomorphism Theorem). *Let* $\mathcal{F}'=\{F_i\}_{i\in I}$

be a finite oriented network with internal tubular neighbourhood $\mathcal{T}'$ embedded in a smooth manifold M. There exists then an isomorphism between the de Rham cohomology with compact supports of some open neighbourhood of $|\mathcal{F}'|$ in M and the de Rham homology of $\mathcal{F}'$ as defined in §2. *More precisely the above map $\rho_\# : H^c_p(U) \to H^c_p(\mathcal{F}', \mathcal{T}')$, see* (11), *is a linear isomorphism.*

PROOF. We induct on the number of elements of I. If I has only one element F, then $\rho_\#$ coincides with the isomorphism $f^\# : H^c_p(U) \to H^c_p(F)$ which is the inverse of Thom isomorphism (see chapter 7 of [5]).

Suppose now that the theorem holds for I with less than r elements ($r \geqq 2$) and assume that I has r elements. Choose r maximal in I and consider the four networks $\mathcal{F}' = \{F_i\}_{i\in I}$, $\{F_i\}_{i\in I-\{r\}}$, $\{F_i\}_{i\leq r}$, $\{F_i\}_{i<r}$.

We have the commutative diagram

$$
\text{(14)}\qquad
\begin{array}{ccccccc}
0 \to A^c_*\Big(\bigcup\limits_{i<r} U_i\Big) & \to & A^c_*\Big(\bigcup\limits_{i\in I-\{r\}} U_i\Big) \oplus A^c_*\Big(\bigcup\limits_{i\leq r} U_i\Big) & \to & A^c_*\Big(\bigcup\limits_{i\in I} U_i\Big) \to 0 \\
\Big\downarrow \rho^* & & \Big\downarrow \rho^* \oplus \rho^* & & \Big\downarrow \rho^* \\
0 \to A^c_*(\{F_i\}_{i<r}) & \xrightarrow[\bar{\varphi}_1]{} & A^c_*(\{F_i\}_{i\in I-\{r\}}) \oplus A^c_*(\{F_i\}_{i\leq r}) & \xrightarrow[\bar{\varphi}_2]{} & A^c_*(\mathcal{F}') \to 0
\end{array}
$$

where we have omitted any reference to the internal tubular neighbourhood which is clear in each case.

The top sequence of (14) is the usual short exact sequence giving rise to the Mayer–Vietoris sequence in de Rham cohomology with compact support. The maps $\bar{\varphi}_1$, $\bar{\varphi}_2$ are induced by

$$\varphi_1 : \bigoplus_{i<r} A^c_p(F_i) \to \bigoplus_{i\in I-\{r\}} A^c_p(F_i) \oplus \bigoplus_{i\in I} A^c_p(F_i),$$

$$\varphi_2 : \bigoplus_{i\in I-\{r\}} A^c_p(F_i) \oplus \bigoplus_{i\leq r} A^c_p(F_i) \bigoplus_{i\in I} A^c_p(F_i),$$

given by

$$\varphi_1((\Phi_i)_{i<r}) = ((\Phi'_i)_{i\in I-\{r\}}, (\Phi'_i)_{i\leq r})$$

where $\Phi'_i = \Phi_i$ if $i < r$ and $\Phi'_i = 0$ otherwise;

$$\varphi_2((\Phi_i)_{i\in I-\{r\}}, (\Psi_i)_{i\leq r}) = (\Omega_i)_{i\in I}$$

where $\Omega_i = \Phi_i - \Psi_i$ if $i < r$, $\Omega_i = \Phi_i$ if $i \in I$ but $i \nleqq r$ and $\Omega_r = -\Psi_r$.

To show that the bottom sequence of (14) is exact, everything is clear except that $\bar{\varphi}_1$ is injective, which is a consequence of Lemma (12) and that $\ker \bar{\varphi}_2 \subset \operatorname{Im} \bar{\varphi}_1$.

To check this last point let us assume that $\varphi_2(\hat{\Phi}, \hat{\Psi}) = \Omega \in J_p^c(\mathcal{F})$. Then, r being maximal, we have $\Omega = \Phi + \Psi$ with $\Phi \in J_p^c(\{F_i\}_{i \in I - \{r\}})$ and $\Psi \in J_p^c(\{F_i\}_{i \leqq r})$.

Set $\Phi' = \hat{\Phi} - \Phi$, $\Psi' = \hat{\Psi} - \Psi$. Thus Φ' and Φ represent the same class α in $A_p^c(\{F_i\}_{i \in I - \{r\}})$, while Ψ' and $\hat{\Psi}$ represent the same class β in $A_p^c(\{F_i\}_{i \leqq r})$. But we have now $\varphi_2(\Phi', \Psi') = 0$. In particular, if we denote the components of Φ' by Φ'_i, $i \in I - \{r\}$, and the components of Ψ' by Ψ'_i, $i \leqq r$, we have $\Phi'_i = \Psi'_i$ for all $i < r$, $\Phi'_i = 0$ for all $i \in I - I_{\leqq r}$ and $\Psi'_r = 0$. Hence if γ is the class represented by $(\Phi'_i)_{i<r} \in A_p^c(\{F_i\}_{i<r})$ we have $\bar{\varphi}_1(\gamma) = (\alpha, \beta)$.

Finally, the corresponding commutative diagram in cohomology, obtained from (14), together with induction hypothesis and the 5-lemma finishes the proof of our theorem.

(15) Similarly, let $A_p^F(U)$ be the $(n-p)$-forms of M whose support is contained in U (i.e. $\Phi \in A_p^F(U)$ if $\Phi \in A^{n-p}(M)$ and $\operatorname{supp} \Phi \subset U$).

Define $\rho : A_p^F(U) \to A_p(\mathcal{F}', \mathcal{T}')$ as follows: if $\Phi \in A_p^F(U)$, choose $\Phi_i \in A_p^F(U_i)$, $i \in I$, such that $\Phi = \Sigma_{i \in I} \Phi_i$ (this can be done easily by using a smooth partition of unity subordinate to the open covering $\{U_i\}_{i \in I}$ of U).

Define then $\rho(\Phi) = [(f_i \Phi_i)_{i \in I}] \in A_p(\mathcal{F}', \mathcal{T}')$ and, as before, it is proved by analogous arguments that ρ is well defined and induces an isomorphism

$$\rho_* : H_p^F(U) \xrightarrow{\cong} H_p(\mathcal{F}', \mathcal{T}'). \tag{16}$$

§4. The canonical isomorphism $H_*(|\mathcal{F}|; R) \to H_*^c(\mathcal{F}, \mathcal{T})$

In this section we prove that de Rham homology of an oriented network $\mathcal{F}$ admitting an internal tubular neighbourhood is isomorphic to the singular homology of $|\mathcal{F}|$ with real coefficients.

Assume first that $\mathcal{F} = \{F_i\}_{i \in I}$ is a finite oriented network and regard it as embedded in some R^N such that $\mathcal{F} \cup \{R^N\}$ is a network with $F_i \subset R^N$ for all $i \in I$.

Let $\tilde{\mathcal{T}}$ be an internal tubular neighbourhood of $\mathcal{F} \cup \{R^N\}$ and let $\mathcal{T}$ be the corresponding induced neighbourhood of $\mathcal{F}$.

Define, for each p,

$$\varphi : H_p(|\mathcal{F}|; R) \to H_p^c(\mathcal{F}, \mathcal{T})$$

to be the linear map such that the following diagram commutes:

$$\begin{array}{ccc} H_p(U;R) & \xleftarrow[\cong]{(-1)^{p(n-p)}\cdot D} & H_c^{n-p}(U;R) \\ & & \Big\uparrow \cong \quad \text{de Rham isomorphism} \\ \Big\uparrow j_\# & & H_p^c(U) \\ & & \Big\downarrow \cong \quad p_\# \ \text{(isomorphism of Theorem 13)} \\ H_p(|\mathcal{F}|;R) & \xrightarrow[\varphi]{} & H_p^c(\mathcal{F},\mathcal{T}) \end{array} \tag{17}$$

where D is Poincaré duality and $j_\#$ is induced in homology with real coefficients by the inclusion $j:|\mathcal{F}|\to U$.

(18) Proposition. *$\varphi : H_p(|\mathcal{F}|;R)\to H_p^c(\mathcal{F},\mathcal{T})$ is an isomorphism.*

Proof. To prove the proposition we show that $H_p(|\mathcal{F}|;R)\to H_p(U;R)$ is an isomorphism for all p, by induction on the number of elements of I. We assume that all manifolds F_i in the family are distinct.

If I has only one element F, the proposition is clear because F is then a strong deformation retraction of U.

Assume that $j_\#$ is an isomorphism for I with at most $r-1$ elements and suppose that I has r elements.

Choose r maximal in I. We have a commutative diagram of chain complexes with exact row sequences

$$\begin{array}{ccccccc} 0\to C_*\Big(\bigcup\limits_{i<r} F_i\Big) & \to & C_*\Big(\bigcup\limits_{i\in I-\{r\}} F_i\Big)\oplus C_*(F_r) & \to & C_*(|\mathcal{F}|;\mathcal{U}) & \to 0 \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow & \\ 0\to C_*\Big(\bigcup\limits_{i<r} U_i\Big) & \to & C_*\Big(\bigcup\limits_{i\in I-\{r\}} U_i\Big)\oplus C_*(U_r) & \to & C_*\Big(\bigcup\limits_{i\in I} U_i;\mathcal{U}'\Big) & \to 0 \end{array}$$

where

$$\mathcal{U}=\Big\{\bigcup_{i\in I-\{r\}} F_i, F_r\Big\},\qquad \mathcal{U}'=\Big\{\bigcup_{i\in I-\{r\}} U_i, U_r\Big\}.$$

Then, the 5-lemma applied to the corresponding commutative diagram in homology shows that $H_*(|\mathcal{F}|;\mathcal{U})\to H_*(\bigcup_{i\in I}U_i;\mathcal{U}')$ is an isomorphism.

Consider next the commutative diagram

$$\begin{array}{ccc} H_*(|\mathcal{F}|;\mathcal{U}) & \xrightarrow{\cong} & H_*\left(\bigcup_{i\in I} U_i;\mathcal{U}'\right) \\ \downarrow & & \downarrow \cong \\ H_*(|\mathcal{F}|;R) & \xrightarrow[i_*]{} & H_*\left(\bigcup_{i\in I} U_i;R\right) \end{array}$$

Set $X_1=\bigcup_{i\in I-\{r\}} F_i$, $X_2=F_r$, $X=|\mathcal{F}|$. We have to show that $H_*(|\mathcal{F}|;\mathcal{U})\to H_*(|\mathcal{F}|;R)$ is an isomorphism, i.e. that the pair (X_1,X_2) is excisive (see [8]), which is the same as proving that the excision map $(X_2,X_1\cap X_2)\subset(X,X_1)$ induces an isomorphism in singular homology (see theorem 4, section 6, chapter 4 of [8]) or equivalently an isomorphism of singular cohomology, since we are using real coefficients.

Recall next the following definition (cf. [3], 6.13 chapter 8). Let Y be a topological space and Y_1, Y_2 subspaces of Y. We say that $Y_1\cap Y_2$ *separates* Y_1, Y_2 if Y_1-Y_2 and Y_2-Y_1 are both open in $Y_1\cup Y_2-Y_1\cap Y_2$.

Observe that X_1 and X_2 are separated by $X_1\cap X_2$ and this implies that (X_1,X_2) is Čech excisive, i.e. $\check{H}(X,X_1)\xrightarrow{\cong}\check{H}(X_2,X_1\cap X_2)$, see 6.15 chapter 8 of [3].

Consider then the commutative diagram

$$\begin{array}{ccc} \check{H}(X,X_1) & \xrightarrow{\cong} & \check{H}(X_2,X_1\cap X_2) \\ \downarrow & & \downarrow \\ H^*(X,X_1) & \longrightarrow & H^*(X_2,X_1\cap X_2) \end{array}$$

The vertical arrows are isomorphisms because $|\mathcal{F}|$ is an Euclidean Neighbourhood Retract, see [3], prop. 6.12, chapter 8. Therefore the lower horizontal arrow is also an isomorphism. This finishes the proof of Proposition (17).

(19) PROPOSITION. *The linear map* $\varphi: H_p(|\mathcal{F}|;R)\to H_p^c(\mathcal{F},\mathcal{T})$, *defined by the commutative diagram* (17), *does not depend on the choices of the embedding of* $\mathcal{F}$ *in* R^N *and that of the internal tubular neighbourhood* $\tilde{\mathcal{T}}$ *inducing the given internal tubular neighbourhood* $\mathcal{T}$ *of* $\mathcal{F}$. *Thus* φ *is canonically associated to* $(\mathcal{F},\mathcal{T})$.

PROOF. We prove this proposition through 6 steps labelled (a),…,(f).

(a) Define $H_*^{\mathrm{dif}}(\mathcal{F};R)$ (resp. $H^*_{\mathrm{dif}}(\mathcal{F};R)$) as the real singular homology (resp. cohomology) of $|\mathcal{F}|$ obtained by using only smooth simplexes ($\sigma:\Delta^p\to|\mathcal{F}|$ is smooth if the image of σ is contained in some F_i of $\mathcal{F}$ and $\sigma:\Delta^p\to F_i$ is smooth in the usual sense).

We have canonical isomorphisms $H_*^{\mathrm{dif}}(\mathcal{F}; R) \xrightarrow{\cong} H_*(|\mathcal{F}|; R)$, $H^*(|\mathcal{F}|; R) \xrightarrow{\cong} H^*_{\mathrm{dif}}(\mathcal{F}; R)$ as can be proved by standard arguments and the following diagram is commutative:

$$\begin{array}{ccc} H^*(|\mathcal{F}|; R) & \xrightarrow{\cong} & H^*_{\mathrm{dif}}(\mathcal{F}; R) \\ \downarrow \cong & & \downarrow \cong \\ H_*(|\mathcal{F}|; R)^* & \xrightarrow[\cong]{} & H_*^{\mathrm{dif}}(\mathcal{F}; R)^* \end{array}$$

(b) Define $\bar{\varphi}$ by the commutative diagram

$$\begin{array}{ccc} H_*^{\mathrm{dif}}(\mathcal{F}; R) & \xrightarrow{\cong} & H_*(|\mathcal{F}|; R) \\ {\scriptstyle \bar{\varphi}} \searrow & & \swarrow {\scriptstyle \cong\ \varphi} \\ & H_*^c(\mathcal{F}, \mathcal{T}) & \end{array}$$

(c) Define a *de Rham cohomology* of a network $\mathcal{F}$ as follows (see [4]): A smooth p-form on $\mathcal{F}$ consists of a family of p-forms $(\Phi_i)_{i \in I}$ such that each Φ_i is a smooth p-form on F_i such that the restriction of Φ_j to F_i coincides with Φ_i (i.e. $\varphi_{ij}^* \Phi_j = \Phi_i$ whenever $i < j$).

Let $A^p(\mathcal{F})$ be the set of this p-form on $\mathcal{F}$. $A^*(\mathcal{F}) = \bigoplus_{p \geq 0} A^p(\mathcal{F})$ is clearly a differential graded algebra with the usual derivative and product of forms. Thus we have a cohomology $H^*_{\mathrm{dR}}(\mathcal{F})$, the de Rham cohomology of $\mathcal{F}$.

(d) Next we prove a generalized de Rham isomorphism theorem $H^*_{\mathrm{dR}}(\mathcal{F}) \xrightarrow{\cong} H^*_{\mathrm{dif}}(\mathcal{F}; R)$ given as follows:

Define $A^p(\mathcal{F}) \xrightarrow{f} C^p_{\mathrm{dif}}(\mathcal{F})$ by

$$f((\Phi_i)_{i \in I})(\sigma) = \int_\sigma \Phi_i \qquad \text{if Image of } \sigma \subset F_i.$$

The map f is clearly well defined and commutes with the exterior derivative by Stokes theorem. Thus f induces a linear map in cohomology $H^*_{\mathrm{dR}}(\mathcal{F}) \to H^*_{\mathrm{dif}}(\mathcal{F}; R)$.

To prove that $f^\#$ is a linear isomorphism we induct on the number of elements of I. If I has only one element F, w' have $H^*_{\mathrm{dR}}(\mathcal{F}) = H^*_{\mathrm{dR}}(F)$, $H^*_{\mathrm{dif}}(\mathcal{F}) = H^*_{\mathrm{dif}}(F)$ and $f^\#$ is de Rham isomorphism.

Suppose that $f^\#$ is an isomorphism for I with at most $r - 1$ elements ($r \geqq 2$) and I has now r elements.

Choose F_r maximal in the family $\mathcal{F}$ and let $\mathcal{F}_1 = \{F_i\}_{i \in I - \{r\}}$, $\mathcal{F}_2 = \{F_i\}_{i < r}$.

The following is a short exact sequence of cochain complexes:

$$0 \to A^*(\mathscr{F}) \xrightarrow{f_1} A^*(\mathscr{F}_1) \oplus A^*(F_r) \xrightarrow{f_2} A^*(\mathscr{F}_2) \to 0$$

where

$$f_1((\Phi_i)_{i\in I}) = ((\Phi_i)_{i\in I-\{r\}}, \Phi_r)$$

and

$$f_2((\Phi_i)_{i\in I-\{r\}}, \Phi_r) = (\Phi_i - \Phi_r \mid_{F_i})_{i<r}.$$

The nontrivial fact is the surjectivity of f_2 and this is done by showing that any differential form on $\mathscr{F}_2$ can be extended to a differential form on F_r.

Then the standard comparison argument using the 5-lemma and induction hypothesis finishes the proof.

(e) Define a linear map $H^p_{\mathrm{dR}}(\mathscr{F}) \xrightarrow[D^p]{} H^c_p(\mathscr{F}, \mathscr{T})^*$ as follows:

Define first a bilinear map

$$A^p(\mathscr{F}) \times A^c_p(\mathscr{F}, \mathscr{T}) \to R$$

by

$$\langle \alpha, \beta \rangle = \sum_{i\in I} \int_{F_i} \Phi_i \wedge \Phi'_i$$

where $(\Phi_i)_{i\in I} = \alpha$ and $(\Phi'_i)_{i\in I}$ represents β.

The map $\langle\ ,\ \rangle$ is well defined and satisfies

$$\langle d\alpha, \beta \rangle + (-1)^p \langle \alpha, \bar{d}\beta \rangle = 0.$$

Thus it induces a bilinear mapping

$$H^p_{\mathrm{dR}}(\mathscr{F}) \times H^c_p(\mathscr{F}, \mathscr{T}) \to R$$

and in particular we obtain a linear map

$$D^p : H^p_{\mathrm{dR}}(\mathscr{F}) \to H^c_p(\mathscr{F}, \mathscr{T})^*.$$

(f) The following diagram commutes, as can be checked easily:

$$\begin{array}{ccc} H^p_{\mathrm{dR}}(\mathscr{F}) & \xrightarrow{\cong} & H^p_{\mathrm{dif}}(\mathscr{F}; R) \\ {\scriptstyle D^p}\downarrow & & \downarrow{\scriptstyle \cong} \\ H^c_p(\mathscr{F}, \mathscr{T})^* & \xrightarrow[\bar{\varphi}^*]{\cong} & H^{\mathrm{dif}}_p(\mathscr{F}; R)^* \end{array}$$

In particular D^p is an isomorphism (this is a generalized Poincaré isomorphism) and $\bar{\varphi}^*$ (and hence φ) is canonically defined.

(20) THEOREM. *If $\mathcal{F}$ is any oriented network admitting an internal tubular neighbourhood $\mathcal{T}$, we have a canonical linear isomorphism* $H_*(|\mathcal{F}|; R) \xrightarrow{\cong} H^c_*(\mathcal{F}, \mathcal{T})$.

PROOF. For any finite subset J of I let $\mathcal{F}_J$ be the corresponding finite subnetwork of $\mathcal{F}$ and denote by $\mathcal{T}_J$ the internal tubular neighbourhood induced by $\mathcal{T}$ on $\mathcal{F}_J$.

We have canonical isomorphisms (18 and 19)

$$\varphi_J : H_p(|\mathcal{F}_J|; R) \xrightarrow{\cong} H^c_p(\mathcal{F}_J, \mathcal{T}_J)$$

and if $J \subset J'$ the following diagram is commutative:

$$\begin{array}{ccc} H_p(|\mathcal{F}_J|; R) & \xrightarrow[\varphi_J]{\cong} & H^c_p(\mathcal{F}_J, \mathcal{T}_J) \\ \Big\downarrow {\scriptstyle \text{induced by inclusion}} & & \Big\downarrow {\scriptstyle \rho^J_{J'}} \\ H_p(|\mathcal{F}_{J'}|; R) & \xrightarrow[\varphi_{J'}]{\cong} & H^c_p(\mathcal{F}_{J'}, \mathcal{T}_{J'}) \end{array}$$

where $\rho^J_{J'}$ sends the class represented by $[(\Phi_i)_{i \in J}]$ to $[(\Phi_i)_{i \in J}; 0_{i \in J'-J}]$.

Finally we finish the proof by observing that

$$H_p(|\mathcal{F}|; R) = \varinjlim_{\substack{J \subset I \\ J \text{ finite}}} (|\mathcal{F}_J|; R) \quad \text{and} \quad H^c_p(\mathcal{F}; \mathcal{T}) = \varinjlim_{\substack{J \subset I \\ J \text{ finite}}} H^c_p(\mathcal{F}_J, \mathcal{T}_J).$$

(21) If we consider $H_*(\mathcal{F}, \mathcal{T})$ instead of $H^c_*(\mathcal{F}, \mathcal{T})$ one could prove, for a finite network $\mathcal{F}$, the existence of an isomorphism

$$H_p(\mathcal{F}, \mathcal{T}) \xrightarrow{\cong} H^p_c(|\mathcal{F}|)^*.$$

§5. Comparison with other constructions and applications

If $\mathcal{F}$ is a network, then $C^\infty(|\mathcal{F}|)$ consists of the real continuous functions $f : |\mathcal{F}| \to R$ such that $f\varphi_i$ is smooth (with the usual meaning) and if U is any open set of $|\mathcal{F}|$, $C^\infty(U)$ are those functions which locally coincide with functions of $C^\infty(|\mathcal{F}|)$. Thus we obtain a sheaf of functions on $|\mathcal{F}|$ and this clearly defines a differentiable space structure on $|\mathcal{F}|$ in the sense of Mostow, see [6].

Furthermore, if $f : \mathcal{F} \to \mathcal{F}'$ is a smooth map as defined in this paper, see §2, then $|f|:|\mathcal{F}|\rightarrow|\mathcal{F}'|$ is a smooth map of differentiable spaces. Therefore we obtain a covariant functor from the category of networks to the category of differentiable spaces.

The associate space $|\mathcal{F}|$ of a network $\mathcal{F}$ has the following topological properties, as can be proved by standard geometric arguments:

(a) $|\mathcal{F}|$ admits a CW complex structure. In particular it is paracompact.

(b) Each point $x\in|\mathcal{F}|$ has an open neighbourhood homeomorphic to a linear network (see example (f) of 4). In particular $|\mathcal{F}|$ is locally smoothly contractible.

Furthermore, $|\mathcal{F}|$ admits smooth partitions of unity subordinate to any locally finite open covering. Therefore we conclude that de Rham cohomology of $|\mathcal{F}|$, as defined by Mostow, is isomorphic to the real singular cohomology of $|\mathcal{F}|$, see theorem 5.2 of [6]. On the other hand, one can show that Mostow's de Rham cohomology for $|\mathcal{F}|$ is canonically isomorphic to de Rham cohomology of $\mathcal{F}$ as defined in this paper; see step (c) in the proof of Proposition 19.

Contrary to most constructions based on differential forms that appear in the literature, which have the goal of obtaining the real singular cohomology of some spaces (see for instance [2], [6], [7] and [9]), our main interest in this paper has been on building a "de Rham homology," rather than cohomology, to be applied to the following situation:

Let G be a torus acting smoothly from the left on a smooth principal bundle $\mathcal{P}:P\rightarrow M$ with structure group K, where M is compact and oriented. The "singular set" A of the action of G on M consisting of those points in M left fixed by some nontrivial subtorus of G (as well as the set A' of points in M left fixed by some nontrivial subgroup of G) is the space associated to a finite oriented network $\mathcal{F}$ (resp. $\mathcal{F}'$), see [4]. In fact, if H is a subgroup of G, define $\tilde{H}$ by $\tilde{H}=\{a\in G \mid ax=x \text{ for all } x\in F_H\}$, where F_H denotes the fixed point set of the action of H on M.

Let I' be the family of all nontrivial subgroups H of G such that $H=\tilde{H}$ and let I be the family of nontrivial subtori H of G appearing as a 1-component of some member of I'.

Order I' and I by $H_1\leqq H_2 \Leftrightarrow H_1\supset H_2$ and we obtain almost directed sets, see definition (1). The families $\mathcal{F}'=\{F_H\}_{H\in I'}$ and $\mathcal{F}=\{F_H\}_{H\in I}$ are finite oriented networks with $|\mathcal{F}|=A$, $|\mathcal{F}'|=A'$. In fact, property (a) of definition (1) is well known, property (b) is easy to check. To show that (c) holds we endow M with a G-invariant Riemannian metric and use the following fact (see lemma 15 of [4]): If F_1 and F_2 are totally geodesic submanifolds of a Riemannian manifold M and $F_1\cap F_2$ is also a submanifold of M, then $T_x(F_1\cap F_2)=T_x(F_1)\cap T_x(F_2)$,

$x \in F_1 \cap F_2$. Property (d) is easily verified since we may restrict ourselves to consider an Euclidean vector space with an orthogonal action of G.

It is well known that the normal bundles of F_H are orientable and therefore F_H is orientable, since M is.

Finally it is clear that $|\mathcal{F}| = A$, $|\mathcal{F}'| = A'$.

(22) *An Application*

Choose a G-invariant principal connection form ω on $\mathcal{P}$, such that ω vanishes on the fundamental vector fields of the action of G on P outside some closed neighbourhood of A in U, where U is the open neighbourhood of A appearing in Theorem (13). Observe that by using this connection form in the Chern–Weil construction, we obtain (as for the Bott's vanishing theorem, [1]) differential forms of degree $2p > n - \dim G$ whose support is contained in U and such that these forms represent characteristic classes of $\mathcal{P}$. Therefore we obtain a homomorphism $S^p(\mathbf{K})_I \to H_c^{2p}(U)$ for $2p > n - \dim G$, where $S^p(\mathbf{K})_I$ denotes the p-linear Ad-invariant forms on the Lie algebra $\mathbf{K}$ of K.

If we compose the above homomorphism with the generalized Thom isomorphism $\rho_* : H_c^{2p}(U) \xrightarrow{\cong} H_{n-2p}^c(\mathcal{F}, \mathcal{T}) \cong H_{n-2p}(A)$, see (13), we have a linear map

$$S^p(\mathbf{K})_I \to H_{n-2p}^c(\mathcal{F}, \mathcal{T}) \qquad \text{for all } 2p > n - \dim G.$$

The above map gives the "residues" at A of the characteristic classes of $\mathcal{P}$ of degree $2p > n - \dim G$, and allows to give in certain cases a differential-form formulation of these residues.

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